

Sieves for Low Autocorrelation Binary Sequences

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Abstract—The value $2e^2$ is calculated by approximations as a conjectured asymptotic limit for the best central to sidelobe energy ratio of very long low autocorrelation binary sequences. The same bound is calculated by approximation for skewsymmetric sequences, hence it is concluded that the requirement of skewsymmetry constitutes a powerful and effective sieve for the search for near-optimal sequences. A second sieve based on the use of complementary sequences is discussed and shown to be quite effective for sequence lengths up to 31, and its investigation for longer lengths by means of computers is suggested. A third and a fourth sieve, based on the selection of restricted classes of complementary sequences, are also discussed.

I. INTRODUCTION

TWO CRITERIA of goodness have been used for low autocorrelation binary sequences. The first is minimal peak sidelobes and has been adopted by Boehmer [1], Turyn [2], Schroeder [3] and others. The second [4], which will be used here, is the "merit factor" F , defined for the binary sequence $\alpha_1, \alpha_2, \dots, \alpha_N$ by

$$F = \frac{N^2}{2 \sum_{i=1}^{N-1} \gamma_i^2}, \quad \text{where } \gamma_i = \sum_{k=i+1}^N \alpha_k \alpha_{k-i}. \quad (1)$$

This second criterion has greater analytical tractability and constitutes an accurate measure of self-generated clutter in applications such as spread spectrum techniques, but it is more demanding on computer time than the first. This and additional criteria were used by Lindner [5].

In the discussion which follows, a conjectured upper bound will be established for F that will be valid for very long binary sequences. This upper bound will be shown to be the same for skewsymmetric sequences, wherefrom it will be inferred that the requirement of skewsymmetry constitutes a useful sieve for narrowing computer searches for long sequences. Three additional sieves based on the use of complementary sequences will be suggested and discussed.

II. DISCUSSION

Consider first a random binary sequence $\alpha_1, \alpha_2, \dots, \alpha_N$ of elements of value $+1$ or -1 (abbreviated to 1 and $-$ where convenient) and the off-peak autocorrelative terms γ_i defined in (1). As the α_k are random, the $N-1$ γ_i will be random also, ranging from $-(N-i)$ to $N-i$, and will have average mean square value

$$\overline{\gamma_i^2} = N-i. \quad (2)$$

While each γ taken by itself is random, there will be constraints among the several γ since each family of γ will have a single α sequence as its source. These constraints will be ignored in what follows, and the several γ derived from any one sequence will be treated as if they were actually independent. The assumption underlying this procedure could be termed the assumption of "mathematical ergodicity" by similarity with the ergodic assumption, which is so useful in statistical mechanics. It is the same assumption that permits calculation of the average distance between consecutive prime numbers of magnitude p as $\ln p$, and it can be expected to be increasingly valid for longer and longer binary sequences.

If, accordingly, we add the various γ given by (2), we find for the average F value of a random sequence:

$$F = \frac{N^2}{2 \sum_{i=1}^{N-1} \overline{\gamma_i^2}} = \frac{N^2}{N(N-1)} \cong 1, \quad (3a)$$

for large N .

We now address ourselves to the problem of finding the highest possible value of F , or lowest value of $\sum \gamma_i^2$, among all binary sequences of length N . For this purpose, we note first that for any one large $N-i$, the discrete distribution, which is given by the binomial coefficients of $(1+1)^{N-i-2-N+i}$ centered at $\gamma_i = 0$, can be approximated by the continuous probability distribution

$$P(\gamma_i) = \frac{1}{\sqrt{2\pi(N-i)}} \exp \left[-\frac{\gamma_i^2}{2(N-i)} \right], \quad -\infty < \gamma_i < \infty$$

or, writing $\gamma_i^2 = E_i$,

$$P(E_i) = \frac{1}{\sqrt{2\pi(N-i)E_i}} \exp \left[-\frac{E_i}{2(N-i)} \right]. \quad (4)$$

Next, we must determine, near the origin, the overall probability distribution of $E = \sum E_i$ which is given by the convolution of all $P(E_i)$ and can be written

$$P(E) = P(E_1) * P(E_2) * \dots * P(E_{N-1}).$$

This convolution can be done conveniently by first taking the successive $P(E_i)$ in pairs, $P(E_{2s-1})$ and $P(E_{2s})$ taken together, and writing

$$\begin{aligned} P(E_{2s-1}) &= \frac{1}{\sqrt{2\pi(N-2s+1)}} \exp \left[-\frac{E_{2s-1}}{2(N-2s+1)} \right] \\ &\cong \frac{1}{\sqrt{2\pi(N-2s+1/2)}} \exp \left[-\frac{E_{2s-1}}{2(N-2s+1/2)} \right] \end{aligned}$$

and

$$P(E_{2s}) = \frac{1}{\sqrt{2\pi(N-2s)E_{2s}}} \exp \left[-\frac{E_{2s}}{2(N-2s)} \right] \\ \cong \frac{1}{\sqrt{2\pi(N-2s+1/2)E_{2s}}} \exp \left[-\frac{E_{2s}}{2(N-2s+1/2)} \right]$$

and convolving such pairs to form

$$P(E_{2s-1} + E_{2s}) = \frac{1}{\sqrt{2\pi(N-2s+1/2)E_{2s-1}}} \\ \cdot \exp \left[-\frac{E_{2s-1}}{2(N-2s+1/2)} \right] * \frac{1}{\sqrt{2\pi(N-2s+1/2)E_{2s}}} \\ \cdot \exp \left[-\frac{E_{2s}}{2(N-2s+1/2)} \right] = \frac{1}{2(N-2s+1/2)} \\ \cdot \exp \left[-\frac{(E_{2s-1} + E_{2s})}{2(N-2s+1/2)} \right].$$

The integral of the probability distribution of E near the origin is of interest here and can be derived from the integral of the convolution of n probability distributions of the form $1/a_i \exp(-x/a_i)$ which is given formally by

$$\int_0^x \frac{1}{a_1} \exp \left(-\frac{x}{a_1} \right) * \frac{1}{a_2} \\ \cdot \exp \left(-\frac{x}{a_2} \right) * \cdots * \frac{1}{a_n} \exp \left(-\frac{x}{a_n} \right) dx \\ = \int_0^x \sum_{i=1}^n \frac{a_i^{n-2}}{\prod_{\substack{s=1 \\ s \neq i}}^n (a_i - a_s)} \exp \left(-\frac{x}{a_i} \right) dx \\ = \frac{x^n}{\prod_{i=1}^n a_i} \sum_{s=0}^{\infty} \frac{(-x)^s}{(n+s)!} \sum \frac{1}{a_i^{b_i} a_j^{b_j} a_k^{b_k} \cdots}, \quad (5)$$

$$b_i + b_j + b_k + \cdots = s, \quad i \leq j \leq k \leq \cdots$$

Two analytically tractable approximations to (5) will be made. The first, and simplest, consists in limiting the sum on s in (5) to the $s = 0$ term. When this is done, we obtain eventually for $\int P(E) dE$, treating N as if it were odd and making a few additional slight approximations along the way,

$$\int_0^E P(E) dE \cong \int_0^E \frac{1}{2N-3} \exp \left[-\frac{(E_1 + E_2)}{2N-3} \right] * \frac{1}{2N-7} \exp \left[-\frac{(E_3 + E_4)}{2N-7} \right] * \cdots * \frac{1}{3} \\ \cdot \exp \left[-\frac{(E_{N-2} + E_{N-1})}{3} \right] dE = \frac{E^{(N-1)/2}}{\frac{N-1}{2}!} \cdot \frac{1}{(3)(7) \cdots (2N-3)} \cong \frac{E^{(N-1)/2}}{\frac{N-1}{2}!} \\ \cdot \frac{1}{\sqrt{(2)(4)(6) \cdots (2N-2)}} = \frac{E^{(N-1)/2}}{\frac{N-1}{2}! \sqrt{2^{N-1}(N-1)!}} \cong \frac{E^{(N-1)/2}}{\left(\frac{N-1}{e}\right)^{N-1} \sqrt{\pi N} (2\pi N)^{1/4}}. \quad (5a)$$

Multiplying this probability integral by the number of nonequivalent sequences of N elements gives the mathematical expectation of the number of sequences having a sidelobe energy between 0 and E . Setting this expression equal to unity and solving for E gives, because of the steepness of the probability curve near the origin, the likely minimal value $E_{\min}$ for a sequence of length N . The number of nonequivalent sequences is obtained by dividing the total number of possible sequences 2^N by the number of sequences which can be derived from one sequence and have the same autocorrelation pattern. This pattern is left unchanged when the sequence is complemented or reversed, and the sign only of the odd-indexed autocorrelation terms is changed when alternate elements are complemented. Thus, with the exception of a small number of symmetric sequences, the 2^N sequences will come in classes of eight which are equivalent. Hence, the total number of nonequivalent sequences is slightly larger than 2^{N-3} and will be approximated by that number.

Now, multiplying on the right in (5a) by 2^{N-3} and setting the product equal to unity yields

$$E_{\min}^{(N-1)/2} \cong \left(\frac{N-1}{e}\right)^{N-1} 2^{-N+3} (2\pi^3 N^3)^{1/4}$$

and

$$E_{\min} \cong \frac{N^2}{4e^2} (512\pi^3 N^3)^{1/(2N)}.$$

Hence, for the theoretical optimal F number, we have

$$F_{\text{th}} = N^2/(2E_{\min}) = 2e^2/(512\pi^3 N^3)^{1/(2N)}. \quad (6)$$

Setting $N = 3, 13, 100$, and 275 in (6) gives the F_{th} values 1.70, 7.58, 13.14, and 14.08, respectively.

The asymptotic F_{th} value for large N , $2e^2 = 14.78$, is just above but intriguingly close to the best known F , namely, $169/12 = 14.08$ for the Barker sequence of length 13. One may be tempted to regard this closeness as coincidence. However, it does raise interest in the question of whether sequences with a figure of merit exceeding that of the Barker sequence of length 13 do in fact exist, even if they were never found.

Our second approximation for (5) consists in replacing $(n+s)!$ in the denominator by the smaller $n!$, and in re-

placing the last summation by the smaller

$$\frac{1}{s!} \left(\sum_{i=1}^n \frac{1}{a_i} \right)^s.$$

This change is equivalent to multiplying on the right of (5a) by

$$1 - E[1/3 + 1/7 + \dots + 1/(2N-3)] \frac{N-1}{2} \\ + E^2[1/3 + 1/7 + \dots + 1/(2N-3)]^2 - \dots \\ = \exp \left[- \frac{1/3 + 1/7 + \dots + 1/(2N-3)}{(N-1)/2} \right].$$

However, since we have with good approximation

$$\frac{1/3 + 1/7 + \dots + 1}{(2N-3)} \cong \frac{1}{4} \log_e [1.481(N-1/2)],$$

the multiplier on the right of (5a) becomes simply

$$[1.481(N-1/2)]^{-E/[2(N-1)]}.$$

Multiplying further by 2^{N-3} , setting this product equal to unity, and introducing a few minor approximations leads eventually to the expression

$$E_{\min}^* (1.481N)^{-E_{\min}^*/N^2} = \frac{N^2}{4e^2} (512\pi^3 N^3)^{1/(2N)}, \quad (7)$$

which determines $E_{\min}^*$ as a function of N . A new theoretical F is then given by

$$F_{\text{th}}^* = N^2/2E_{\min}^*. \quad (6a)$$

Equation (7) admits of solutions only for N in the range from 5 to 35 386. Setting $N = 13, 100$, and 275 ; solving for $E_{\min}^*$ and substituting (6a) gives the F_{th}^* values 5.90, 10.31, and 10.61, respectively. F_{th}^* decreases eventually, and for N equal to 1000, 10 000, and 35 386, F_{th}^* has the values 10.16, 8.24, and 5.44, respectively. Since (7) is unsolvable for $N > 35\,386$, while we know of course that an optimal F value exists for all N , we reach the tentative conclusion that (7) and (6a), if they are more accurate than (6), can only be so in some intermediate range, around $N = 100$, and this leaves open the question of whether $2e^2$ may not in fact constitute a true asymptotic value for F_{opt} when $N \rightarrow \infty$, as would follow from (6).

We come now to the first of the four sieves mentioned above aimed at reducing the computational labor and expense involved in the investigation of these long, low autocorrelation sequences. Unless an algorithm is found that will enable one to make a beeline for the sequences with the highest F value for certain favorable values of N , an eventuality which we consider unlikely, the practical way of finding long sequences with high figures of merit will be sieve, sieve, sieve again, and then search.

The first sieve is derived from the observation that all odd Barker sequences are skewsymmetric, i.e., that for $N = 2n + 1$ these sequences satisfy

$$\alpha_{n+i} = (-1)^i \alpha_{n-i}.$$

This skewsymmetry causes all γ_i to vanish when i is odd so that any skewsymmetric sequence can be considered as the interleaving of a symmetric sequence of the form

$$A \ c \ \bar{A}$$

and an antisymmetric sequence of the form

$$B \ \bar{B}'$$

where the overbar indicates sequence reversal and the prime indicates complementing of the sequence. The length of the B sequence is equal to that of the A sequence when n is even and is one element longer when n is odd, and the center element c can be either $+1$ or -1 .

It is apparent that the autocorrelation γ_{2i} of the complete interleaved sequence is the sums of the autocorrelations of the two "detwined" components for i shifts and that all contributions to γ_{2i} come in pairs, except for a single $+1$ from the symmetric component for every even i or a single -1 from the antisymmetric component for every odd i . We may write, therefore,

$$\overline{\gamma_{2i}^2} = \overline{\gamma_{2i}^2} + 1$$

where every γ_{2i} will be treated as if it were the result of the random addition of $(n-i)$ digits which are $+2$ or -2 so that

$$\overline{\gamma_{2i}^2} = 4(n-i), \quad i = 1, 2, \dots, n.$$

We again invoke the assumption of "mathematical ergodicity" and find for the average F value of a skewsymmetric sequence with random A and B portions,

$$F = \frac{(2n+1)^2}{2 \sum_{i=1}^n \overline{\gamma_{2i}^2}} = \frac{(2n+1)^2}{2 \sum_{i=1}^n \overline{\gamma_{2i}^2} + 1} \\ = \frac{(2n+1)^2}{2 \sum_{i=1}^n [4(n-i) + 1]} = \frac{(2n+1)^2}{4n^2 - 2n} \cong 1 \quad (3b)$$

as was found in (3a) for general sequences.

Next, we determine as before the probability distribution of the γ_{2i} and obtain

$$P(\gamma_{2i}) = \frac{1}{\sqrt{8\pi(n-i)}} \exp \left[- \frac{\gamma_{2i}^2}{8(n-i)} \right], \\ -\infty < \gamma_{2i} < \infty.$$

Thus, the probability distribution of $E'_{2i} = \gamma_{2i}^2$ is

$$P(E'_{2i}) = \frac{1}{\sqrt{8\pi(n-i)E'_{2i}}} \exp \left[- \frac{E'_{2i}}{8(n-i)} \right], \\ 0 < E'_{2i} < \infty.$$

We then obtain the probability distribution of

$$E' = \sum_{i=1}^{n-1} E'_{2i} = \sum_{i=1}^{n-1} \gamma_{2i}^2$$

which is given formally by the convolution:

$$P(E') = P(E'_2) * P(E'_4) * \dots * P(E'_{2n-2})$$

by again taking the $P(E'_{2i})$ in pairs and writing

$$P(E'_{4s-2}) = \frac{1}{\sqrt{8\pi(n-2s+1)E'_{4s-2}}} \cdot \exp \left[-\frac{E'_{4s-2}}{8(n-2s+1)} \right] \cong \frac{1}{\sqrt{8\pi(n-2s+1/2)E'_{4s-2}}} \cdot \exp \left[-\frac{E'_{4s-2}}{8(n-2s+1/2)} \right]$$

and

$$P(E'_{4s}) = \frac{1}{\sqrt{8\pi(n-2s)E'_{4s}}} \exp \left[-\frac{E'_{4s}}{8(n-2s)} \right] \cong \frac{1}{\sqrt{8\pi(n-2s+1/2)E'_{4s}}} \exp \left[-\frac{E'_{4s}}{8(n-2s+1/2)} \right]$$

We then obtain for each pair

$$P(E'_{4s-2} + E'_{4s}) \cong \frac{1}{\sqrt{8\pi(n-2s+1/2)E'_{4s-2}}} \exp \left[-\frac{E'_{4s-2}}{2(n-2s+1/2)} \right] \cdot \frac{1}{\sqrt{8\pi(n-2s+1/2)E'_{4s}}} \exp \left[-\frac{E'_{4s}}{8(n-2s+1/2)} \right] = \frac{1}{8(n-2s+1/2)} \exp \left[-\frac{E'_{4s-2} + E'_{4s}}{8(n-2s+1/2)} \right]$$

We obtain, as before, two approximations for the probability integral $\int P(E') dE'$. The first, consisting in keeping only the term for $s = 0$ in (5), yields

$$\int_0^{E'} P(E') dE' = \frac{E'^{(n-1)/2}}{\frac{n-1}{2}! (12)(28) \cdots [8(n-3/2)]} \cong \frac{E'^{(n-1)/2}}{\frac{n-1}{2}! \sqrt{(8)(16)(24) \cdots [8(n-1)]}} = \frac{E'^{(n-1)/2}}{\frac{n-1}{2}! \sqrt{8^{n-1}(n-1)!}} \cong \frac{E'^{(n-1)/2}}{\left(\frac{n-1}{e}\right)^{n-1} (2\pi^3 n^3)^{1/4}}$$

Multiplying this last expression by the number of nonequivalent skewsymmetric sequences 2^{n-1} , setting the expression obtained equal to unity, and solving for E' yields for that minimal $E'_{\min}$ value, which we have a mathematical expectation unity of finding among all nonequivalent sequences of length $N = 2n + 1$,

$$E'_{\min}^{(n-2)/2} = \left(\frac{n-1}{e}\right)^{n-1} (2\pi^3 n^3)^{1/4}$$

or

$$E'_{\min} = \left(\frac{n-1}{e}\right)^2 (2\pi^3 n^3)^{1/[2(n-1)]} \cong \left(\frac{N}{2e}\right)^2 \left(\frac{\pi^3 N^3}{4}\right)^{1/N} \quad (8)$$

Thus

$$E_{\min} = E'_{\min} + N/2 \cong \left(\frac{N}{2e}\right)^2 \left(\frac{\pi^3 N^3}{4}\right)^{1/N} + \frac{N}{2}$$

and

$$F_{\text{sth}} = \frac{N^2}{2E_{\min}} = \frac{2e^2}{\left(\frac{\pi^3 N^3}{4}\right)^{1/N} + \frac{2e^2}{N}} \quad (9)$$

For the N equal to 3, 13, 100, 275, and 1000, (9) yields the F_{sth} values 1.36, 4.54, 11.22, 13.14, and 14.24, respectively. The second approximation, entirely similar to the second one made earlier for generally asymmetric sequences, results in the introduction of the factor $(1.481N)^{-E'^*/N^2}$ in (8) and gives

$$E'^* (1.481N)^{-E'^*/N^2} = \left(\frac{N}{2e}\right)^2 \left(\frac{\pi^3 N^3}{4}\right)^{1/N} \quad (8a)$$

Thus

$$F_{\text{sth}}^* = \frac{N^2}{2E'^* + N} \quad (9a)$$

Equation (8a) is solvable for N in the range from 3 to 35 245. For $N = 3, 13, 101, 1001, 10\,001$, and 35 245, (8a) together with (9a) yields the F_{sth}^* values 1.01, 3.75, 9.67, 9.25, 8.20, and 5.44, respectively. The same general remarks apply to the solutions of (8a) and (9a) as were made above in connection with those of (7) and (6a), and the issue of the asymptotic value of F for large N remains open.

The great similarity and the near equality of the F_{th} values obtained for general sequences and for skewsymmetric sequences indicates that the requirement of skewsymmetry constitutes a useful sieve to cut down computational labor; for approximately equal cost, skewsymmetric sequences of twice the length with almost the same F_{opt} values can be obtained by computer search.

Our second sieve is based on the general use of complementary sequences (CS) of lengths which are a power of 2. These sequences have been discussed at length in [6] to which reference is made for proofs, but a brief description of the algorithm used for producing them, as well as a suggested mode of coding, may be useful here and will be given below.

These sequences are based on the use of simply periodic Walsh functions, $(2^n - 1)$ followed by $(2^n + 1)$, etc. For a CS of length 2^5 , for instance, the simply-periodic Walsh function of period 2^5 will be given the index 1, the simply-periodic Walsh function of period 2^4 will be given the index 2, etc. These functions are then written one above the other in a chosen scrambled order, and within each column, a one will be written whenever there is an odd number of pairs of (vertically) adjacent ones above, and a minus written otherwise; this gives the first of a pair of CS's. Multiplication of the elements of this first sequence by the corresponding elements of the Walsh function written just above it gives the second of the pair (multiplication by the first Walsh function written would give the same sequence in the reversed order, and complemented for sequence lengths which are odd powers of 2).

An example is given below for sequences of length 2^5 (the carets are for a subsequent reference).

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3 ---- 1 1 1 1 ---- 1 1 1 1 ---- 1 1 1 1 ---- 1 1 1 1
1 -----
2 ----- 1 1 1 1 1 1 1 1 ----- 1 1 1 1 1 1 1 1
5 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1 - 1
4 -- 1 1 -- 1 1 -- 1 1 -- 1 1 -- 1 1 -- 1 1 -- 1 1 -- 1 1
31254 --- 1 --- 1 - 1 --- 1 ----- 1 1 1 1 - 1 - 1 1 - 1 ---
31254 1 1 - 1 1 1 - 1 1 - 1 1 - 1 - 1 - 1 - 1 - 1 1 1 1 ---
31254 - 1 - - - 1 - - - - 1 - - - 1 1 - 1 1 - 1 - - - - 1 1 1 1 -
31254 1 - - - 1 - - - 1 1 - 1 1 1 - 1 - 1 1 1 1 - - - 1 1 - 1 - 1 -

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The code number 31254 is assigned to the first sequence of the complementary pair, and 31254̄ is assigned to the second sequence, which was obtained by multiplying the first by the Walsh function of index 4. Multiplying both sequences of the pair by any selection of the Walsh functions between the first and the last (of indexes 1 and/or 2 and/or 5) yields other complementary pairs; the second pair shown below the first was obtained by multiplying each sequence of the first pair by the Walsh functions of indices 1 and 5 and, accordingly, are given the code numbers 31254̄ and 31254. The same functions would be obtained by interverting the order 31254 of the Walsh functions chosen, so that it may be stipulated that the first index shown (three in the example above) be always smaller than the last. Accordingly, there will be $(1/2)5!$ ways of ordering the five Walsh functions, and each ordering can give eight distinct pairs of CS's by multiplying, or not multiplying, the two CS's of any pair so obtained by the Walsh functions of intermediate indices (1, 2, and 5 in the example above). Hence, the number of distinct CS's of length 2^n is $2^{n-3}n!$. The complementary property that the sum of the autocorrelations of the two sequences of a pair is zero for all shifts can be easily verified in the two examples above. A complementary pair will be denoted simply by the code for the first sequence of the pair.

The simplest application of CS's to help form low autocorrelation binary sequences consists in choosing a pair of CS's of length 2^n for the A and B sequences of (7a) and (7b), so as to obtain a binary sequence of length $2^{n+2} + 1$. Each pair of CS's can be used in this manner in 16 non-equivalent ways: a one or a minus can be selected for c ; the A sequence, or the B sequence, or both may be reversed, and A and B may be interchanged. Now, when the autocorrelations of the A c $\bar{A}$ and B $\bar{B}$ sequences are formed and added, the autocorrelation of the B $\bar{B}'$ sequence is comprised of the two individual autocorrelations of B and $\bar{B}'$, which are equal, and the crosscorrelation of B and $\bar{B}'$ (hereafter denoted by $B \times \bar{B}'$) which is the negative of $B \times \bar{B}$. Similarly, the autocorrelation of the A c $\bar{A}$ sequence will contain three components: the individual autocorrelations of A and $\bar{A}$ which are equal and, by virtue of the complementary property, are exactly compensated by the individual autocorrelations of B and $\bar{B}'$; the crosscorrelation of A and $\bar{A}$ shifted by one element (hereafter denoted by $A \times 0 \bar{A}$); and the effect of $c = \pm 1$ which is to add or subtract twice the autocorrelation of A from $A \times 0 \bar{A} - B \times \bar{B}$.

The effect of c will be neglected in the evaluation of the average benefit derivable from the use of CS's, so that the essential autocorrelative terms will be the difference $A \times 0 \bar{A} - B \times \bar{B}$. Were these terms random, this would mean an approximate reduction of one half in self-generated noise, so that this method of construction would have the property of yielding F values averaging 2 before any further selection, i.e., we would have traded the enormous number of choices afforded by all skewsymmetric sequences and settled for the much smaller number afforded by the use of CS's for the advantage of an initial gain of 2 in F value. Actually we are doing better, because the difference of the crosscorrelations of A and B with $\bar{A}$ and $\bar{B}$, respectively, are not random. Designate the "energy" of this difference by $E(A \times 0 \bar{A} - B \times \bar{B})$ and assume, e.g.,

$$A = 132 = - - - 1 - 1 - -$$

and

$$B = 13\bar{2} = - - 1 - - 1 1 1$$

We then have

$$A \times 0 \bar{A} = 0 \ 1 \ 2 \ -1 \ 0 \ 1 \ -2 \ 7 \ 0 \ 3 \ -2 \ 1 \ 0 \ 3 \ 2 \ 1$$

$$B \times \bar{B} = 1 \ 2 \ 3 \ 0 \ -3 \ -2 \ -1 \ 0 \ -5 \ -2 \ 5 \ 0 \ -1 \ 2 \ 1$$

$$A \times 0 \bar{A}$$

$$-B \times \bar{B} = -1 \ -1 \ -1 \ -1 \ 3 \ 3 \ -1 \ 7 \ 5 \ 5 \ -7 \ 1 \ 1 \ 1 \ 1$$

and

$$E(A \times 0 \bar{A} - B \times \bar{B}) = 10 \times 1^2 + 2 \times 5^2 + 2 \times 7^2 = 176.$$

Inverting A , or B , or both, would have yielded

$$E(A \times 0 A - B \times \bar{B}) = 208,$$

$$E(A \times 0 \bar{A} - \bar{B} \times B) = 144,$$

$$E(\bar{A} \times 0 A - \bar{B} \times B) = 176,$$

respectively.

(Combining these four arrangements with the effect of $c = \pm 1$ and interchanging A and B would yield the 16 non-equivalent skewsymmetric sequences for the pair of CS's A and B .)

It has been found that these E values are 12 for all CS's of length 2, 40 for those of length 4, 144, 176, or 208 for those of length 8, 544, 608, 672, 736, or 800 for those of length 16, etc. Their average, for CS's of length $2^n = 2s$ or $2^{n+1} = 2s+1$ are

$$\overline{E(2s)} = 2^{2s+1} \frac{2^{2s+2} - 1}{3} \cong \frac{1}{3} 2^{2n+3}$$

$$\overline{E(2s+1)} = 2^{2s+2} \frac{2^{2s+3} + 1}{3} \cong \frac{1}{3} 2^{2n+3},$$

respectively. Thus before any further selection, the F value for skewsymmetric sequences of length $2^{n+2} + 1$ formed therefrom would average

$$F = \frac{(2^{n+2} + 1)^2}{2E(n)} \cong 3.$$

Our third sieve is derived from the fairly considerable fluctuations of the E values. For the six CS's of length 8, there are four E values of 144 given by the appropriate (one out of four) arrangements of the 132, 132, 213, and 213 pairs. For the 48 complementary sequences of length 16, there are four E values of 544 given by the appropriate (one out of four) arrangements of the 2134, 2134, 3214, and 3214 sequence pairs, 48 of value 608, 88 of value 672, 48 of value 736 and four of value 800. The number of low E values among the 1920 values yielded by the 480 complementary pairs of 32 has not been determined, but it has been observed that one each out of the two foursomes yielded by the 32145 and 32145 pairs have the value 2140, which is 512 below the 2652 average for complementary sequences of 32. Thus assuming that the least E values for longer sequences continue to be 2^{2n-1} below the average of E , we should obtain an algorithmic F value of

$$F = \frac{(2^{n+2} + 1)^2}{2 \left(\frac{2^{2n+3}}{3} - 2^{2n-1} \right)} \cong \frac{48}{13} = 3.692 \dots,$$

a figure which happens to be very close to one-fourth of our hypothetical goal of $2e^2$, i.e., $1/2e^2 = 3.694$.

The fourth sieve will be introduced after a discussion of

$$\begin{aligned} 1\bar{2}3 &= --1---1 \\ 1\bar{2}\bar{3} &= -111-1-- \\ 1\bar{2}3 \times 1\bar{2}3 &= 1\ 2\ -1\ 0\ 5\ 2\ 3\ 0\ -1\ 6\ 1\ 0\ -1\ -2\ 1 \\ 1\bar{2}3 \times 000001\bar{2}3 &= 1\ -2\ -1\ 0\ 5\ -2\ 3\ 0\ -1\ 6\ 1\ 0\ -1\ 2\ 1 \\ 1\bar{2}3 \times 1\bar{2}3 - 1\bar{2}\bar{3} \times 000001\bar{2}3 &= 1\ 2\ -1\ 0\ 5\ 1\ 5\ 1\ -1\ 1\ 3\ -3\ -1\ -1\ -5\ 1\ 0\ -1\ 2\ 1 \\ E(1\bar{2}3 \times 1\bar{2}3 - 1\bar{2}\bar{3} \times 000001\bar{2}3) &= 136. \end{aligned}$$

possible modifications of the simple application of CS's described above to form skewsymmetric sequences of length $2^{n+2} + 1$. There are three possible modifications.

a) From the CS's A and B , we may form skewsymmetric sequences longer than $2^{n+2} + 1$ by interleaving symmetric sequences of the form $a\ A\ a_1\ c\ \bar{a}_1\ \bar{A}\ \bar{a}$ and antisymmetric sequences of the form $b\ B\ b_1\ \bar{b}_1\ B'\ \bar{b}'$ where a, a_1, b, b_1 , etc., denote short binary sequences which are either appended to, or inserted in, the basic $A\ c\ \bar{A}$ and $B\ \bar{B}'$ sequences. It is readily seen that the number of new sequences so constructed is much greater than before; if we designate by m the total number of elements in a and a_1 (or b and b_1) the number of these new $4m$ longer sequences is equal to the old number multiplied by $m^{2^{2m}}$, which means an enormously increased search.

b) We may form shorter skewsymmetric sequences by selecting in, e.g., the A sequence, an element about which that sequence is symmetric (as far as it goes), such as the element marked $\hat{1}$ in the 31254 sequence above, and then folding about that element to form a length 57 symmetric

sequence; likewise, a point can be selected in the B sequence about which the short end is antisymmetric to the corresponding portion of the long end, such as the point marked with a caret in the 31254 sequence above, about which this sequence can be folded to form a length 58 antisymmetric sequence that can then be interleaved with the length 57 sequence to form a length 115 skewsymmetric sequence. It will be noted that the sequences shorter than $2^{n+2} + 1$ formed by this b) process are fewer in number: when the c term is dropped and A is folded about one of its end elements to form, $(A\ \bar{A})^*$, and from it and $B\ \bar{B}'$ a length $2^{n+2} - 1$ skewsymmetric sequence, the number of choices is halved; it is halved again on the average when the $B\ \bar{B}'$ sequence is reduced to the two element shorter $(B\ \bar{B}')^*$ sequence; etc.

c) The a) and b) processes may be combined. For instance, leaving $A\ c\ \bar{A}$ alone, we may form a $b\ (B\ \bar{B}')^* \bar{b}'$ sequence in which the number of elements in b and $\bar{b}'$ is equal to, or two more than, the number of elements lost by the antisymmetric folding of $(B\ \bar{B}')^*$ about a point of antisymmetry.

When the c) process is applied, it is clear that the search for favorable $E(A \times 0\ \bar{A} - B \times \bar{B})$ values should be extended to others such as $E(A \times 000\ \bar{A} - B \times B)$ or $E(A \times 0\ \bar{A} - B \times 00\ \bar{B})$, etc. This was done by hand for the 168 possible cases of relatively shifted crosscorrelations of length 8 CS's ($168 = 7 \times 4 \times 6$; 7 for the seven possible odd shifts of $A\ \bar{A}$ with respect to $B\ \bar{B}$; four for the reversal of A or B or both, and $6 = 2^{n-3} n!$ when $n = 3$ for the number of distinct CS's of length 8). A single case was found for which E was smaller than 144 and is shown below.

Two more cases of $E = 144$ were also encountered for a total of 6.

If now we were to select those complementary sequences for which E is a minimum, we would have an extremely selective fourth sieve, for very few minima may be found for E , and these would specify the relative shifts of the individual $A \times \bar{A}$ and $B \times \bar{B}'$ crosscorrelation as well. These minima have not been searched for among the 2880 $A \times \bar{A}$ and $B \times \bar{B}'$ configurations of length 16 complementary sequences [(15 odd shifts) $\times$ (4 for A and/or B reversals) $\times$ (48 possible CS's)]. However, a spot check has revealed the interesting case

$$E(21\bar{3}4 \times 21\bar{3}4 - 21\bar{3}4 \times 000000000000021\bar{3}4) = 440$$

instead of the 672 average. The computer search would be interesting, but even more interesting would be the discovery of a generalizable algorithm for their determination, an eventuality to be considered in view of the intriguing group-theoretical features suggested by these correlations (see Table I).

TABLE I
E DEPARTURES FROM AVERAGE FOR 8-ELEMENT CS

Sequences	1	3	5	7	9	11	13
123	0	-3	-1	-2	-2	3	1
12 $\bar{3}$	0	3	1	2	2	-3	-1
123	(1) 0	0	0	0	0	0	0
12 $\bar{3}$	0	0	0	0	0	0	0
1 $\bar{2}$ 3	0	-3	-5	2	2	-1	1
1 $\bar{2}$ 3	0	3	5	-2	-2	1	-1
1 $\bar{2}$ 3	0	0	0	0	0	0	0
1 $\bar{2}$ 3	0	0	0	0	0	0	0
132	0	1	3	-2	-2	1	-1
13 $\bar{2}$	0	1	-1	2	2	-3	-1
132	-4	1	1	-2	2	-1	-1
13 $\bar{2}$	4	1	1	2	-2	-1	-1
1 $\bar{3}$ 2	0	-1	-3	2	2	-1	1
1 $\bar{3}$ 2	0	-1	1	-2	-2	3	1
1 $\bar{3}$ 2	4	-1	-1	2	-2	1	1
1 $\bar{3}$ 2	-4	-1	-1	-2	2	1	1
213	(3)-4	1	1	-4	0	1	1
21 $\bar{3}$	4	-1	-1	4	0	-1	-1
213	0	0	0	0	0	0	0
21 $\bar{3}$	0	0	0	0	0	0	0
2 $\bar{1}$ 3	4	1	1	0	-4	1	1
2 $\bar{1}$ 3	(2)-4	-1	-1	0	4	-1	-1
2 $\bar{1}$ 3	0	0	0	0	0	0	0
2 $\bar{1}$ 3	0	0	0	0	0	0	0

It is conjectured that this fourth sieve could yield starting F well over 4 for long sequences, beyond which one may be led to searches over all a, a_1, b, b_1 , etc., appended or inserted short sequences, as well as by the exploitation of sequences shortened by the use of symmetries about elements of A or B or antisymmetries about points of B or A .

Table I has been formed from the $E(A \times m \bar{A} - B \times \bar{B})$ values found for the 168 possible cases mentioned above, and shows the departures of E from the 176 average divided by 8; m designates the odd positive or negative shifts of $A \times \bar{A}$ with respect to $B \times \bar{B}$ or $\bar{B} \times B$. The (1), (2), and (3) notations are for subsequent reference.

Table II gives, for length 16 CS's, the figures corresponding to the first column of the preceding table, which are 192 in numbers. Here the departures are divided by 64 and the number of occurrences only are given for each of the 12 families of CS's.

III. APPLICATIONS

The extensive computational effort required for strict application of the sieves discussed above to discover low autocorrelation sequences has not been made yet. As a compromise, the problem has been turned around, and a search has been made of all skewsymmetric sequences up to $N = 59$ with optimal F in order to determine which of these sequences would have been found by an application of these sieves. Table III is a tabulation of the sequences

TABLE II
E DEPARTURES FROM AVERAGE FOR 16-ELEMENT CS

Sequence pair	-2	-1	0	1	2
1234			16		
1243		4	8	4	
1324		4	8	4	
1342		8		8	
1423		4	8	4	
1432		4	8	4	
2134	2	4	-4	4	2
2143		8		8	
23		8		8	
2413			16		
3124		4	8	4	
3214	2		12		2
Total	4	48	88	48	4

$\Sigma=192$

found for the 29 cases 3-59, many of which were multiples, and F_{sth} and F_{sth}^* are the values computed from (9) and from (8a) and (9a), respectively.

An initial remark is that the sequences in Table III form three classes. The first consists of the five odd-numbered Barker sequences with an F -value of the order of thrice F_{sth} . The second class comprises the three sequences for $N = 15, 21$, and 27 with F -values half again as large as F_{sth} . The third class consists of all others which have F -values consistently below F_{sth} , but tending to approach it for large N ; the F/F_{sth} ratio reaching 0.97 for $N = 45$ and 49 and 0.98 for $N = 43$. On the other hand, the F -values of the third group seem to weave at first about F_{sth}^* , but, for $N > 37$, are consistently above F_{sth}^* . This might indicate that the F_{sth} values may be approached by F for larger N , in which case $F = 2e^2$ may be reached asymptotically. However, more analytical and/or computational work is required to determine the value of the right side of (5) for large n , for $a_i = 4i - 1$, and for x in the $0.14n^2$ to $0.2n^2$ range, corresponding to F th values in the 10 to 14 range.

We now consider the hypothesized results of using the sieves. These were estimated by making a survey of the CS's that could be extracted out of the detwined symmetric and antisymmetric components of the skewsymmetric sequences, which have all been indicated in Table III as the sequence portions comprised between asterisks. The cases of CS's of length 2 are trivial, and of length 4 near trivial. Those of length 8, for $N = 21, 23, 27, 29$, and 31 are interesting, however. In all but the 29 case, both CS's are at the extreme end of the detwined components, and all correspond to a contraction of the CS by the use of an element of symmetry, and, except for the $N = 31$ case, of a point of antisymmetry.

With reference to Table I, the numbers in parenthesis indicate when advantage was taken of departures of E from the 176 average, which occurred in four out of the five cases shown. All cases correspond to the first column of Table I and no advantage was taken of the least E value encountered, corresponding to the -5 of the fifth line.

The two disappointing features are the absence of length 8 CS's in the detwined components for $N > 31$ and the absence of length 16 CS's for the largest N values. A halt was called in the search when over one hour of computing time was required for $N = 59$, and the hope is expressed here that a computer Mecerne will extend the search to the sixties to resolve this point. Unless and until this is done,

TABLE III
SKEWSYMMETRIC SEQUENCES WITH OPTIMAL F

	N=3	F=4.50	$F_{sth}=1.36$	$F_{sth}^*=1.01$	$1_1 1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 - 1_1 1_1 1_1$
	$1_1 -$				$1_1 - 1_1 1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 1_1 1_1 - 1_1$
	N=5	F=6.25	$F_{sth}=2.14$	$F_{sth}^*=1.67$	
	$*1_1 1_1^* 1_1^*$				N=35 (2 sol.) F=6.88 $F_{sth}=7.94$ $F_{sth}^*=6.54$
	N=7	F=8.33	$F_{sth}=2.84$	$F_{sth}^*=2.28$	$1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 1_1$
	$*1_1 1_1^* 1_1^* -$				$1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 1_1$
	N=9 (2 sol.)	F=3.38	$F_{sth}=3.47$	$F_{sth}^*=2.83$	
	$*1_1 1_1^* - 1_1 1_1$				N=37 F=6.46 $F_{sth}=8.14$ $F_{sth}^*=6.69$
	$1_1 1_1 1_1 - 1_1 1_1$				N=39 (2 sol.) F=7.68 $F_{sth}=8.32$ $F_{sth}^*=6.83$
	N=11	F=12.10	$F_{sth}=4.04$	$F_{sth}^*=3.32$	$1_1 1_1 1_1 1_1 - 1_1 - 1_1 - 1_1 1_1 - 1_1 1_1 - 1_1 1_1 1_1$
	$*1_1 1_1^* - 1_1^* 1_1^* -$				$1_1 - 1_1 - 1_1 - 1_1 1_1 1_1 1_1 1_1 1_1 1_1 - 1_1 - 1_1 1_1$
	N=13	F=14.08	$F_{sth}=4.54$	$F_{sth}^*=3.75$	
	$*1_1 1_1 1_1^* - 1_1^* 1_1^* 1_1$				N=41 F=7.78 $F_{sth}=8.49$ $F_{sth}^*=6.96$
	N=15	F=7.50	$F_{sth}=5.00$	$F_{sth}^*=4.14$	$1_1 1_1 - - - 1_1 1_1 - 1_1 - 1_1 - 1_1 - - - 1_1 - 1_1 1_1$
	$*1_1 1_1 1_1^* - 1_1^* 1_1^* - 1_1 1_1$				N=43 F=8.48 $F_{sth}=8.66$ $F_{sth}^*=7.09$
	N=17	F=4.52	$F_{sth}=5.42$	$F_{sth}^*=4.49$	$1_1 1_1 1_1 - 1_1 - 1_1 1_1 - - - 1_1 - 1_1 - 1_1 1_1 - - - 1_1 - 1_1$
	$*1_1 1_1^* - - - 1_1^* 1_1^* - 1_1 1_1$				N=45 F=8.58 $F_{sth}=8.81$ $F_{sth}^*=7.20$
	N=19 (2 sol.)	F=5.47	$F_{sth}=5.79$	$F_{sth}^*=4.80$	$1_1 1_1 1_1 1_1 - 1_1 - 1_1 1_1 1_1 - - - 1_1 1_1 - - - 1_1 1_1 1_1$
	$*1_1 1_1^* - - - 1_1^* 1_1^* 1_1 1_1 1_1 -$				N=47 (5 sol.) F=8.18 $F_{sth}=8.96$ $F_{sth}^*=7.31$
	$*1_1 1_1^* - - - 1_1^* 1_1^* - - - 1_1 1_1 -$				$1_1 1_1 1_1 - - - 1_1 1_1 - 1_1 - 1_1 1_1 - - - 1_1 1_1 - 1_1 1_1$
	N=21	F=8.48	$F_{sth}=6.14$	$F_{sth}^*=5.09$	$1_1 1_1 1_1 1_1 - 1_1 1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 1_1 1_1 1_1$
(1)	$*1_1 1_1^* - - - 1_1^* 1_1^* - 1_1 1_1 1_1 -$				$1_1 1_1 1_1 1_1 - 1_1 1_1 - - - 1_1 1_1 1_1 - 1_1 - 1_1 1_1 1_1 1_1$
	N=23 (4 sol.)	F=5.19	$F_{sth}=6.46$	$F_{sth}^*=5.35$	N=49 F=8.83 $F_{sth}=9.10$ $F_{sth}^*=7.42$
(2)	$*1_1 1_1^* - - - 1_1^* 1_1^* 1_1 1_1 1_1^* - 1_1 -$				$1_1 1_1 - 1_1 1_1 - - - 1_1 1_1 1_1 - 1_1 1_1 1_1 1_1 - 1_1 - 1_1 1_1 1_1 1_1$
	$1_1 1_1 - 1_1 1_1 1_1 1_1 - 1_1 - 1_1 -$				N=51 F=8.50 $F_{sth}=9.23$ $F_{sth}^*=7.52$
	$1_1 1_1 1_1 1_1 - 1_1 1_1 - 1_1 1_1 1_1$				$1_1 - - - 1_1 1_1 - 1_1 1_1 1_1 - - - 1_1 - 1_1 - 1_1 1_1 1_1 - 1_1 - 1_1 1_1 1_1$
	$1_1 - - - 1_1 1_1 1_1 - 1_1 1_1$				N=53 F=8.26 $F_{sth}=9.35$ $F_{sth}^*=7.61$
	N=25	F=6.01	$F_{sth}=6.75$	$F_{sth}^*=5.59$	$1_1 - 1_1 1_1 1_1 1_1 1_1 - - - 1_1 1_1 - 1_1 - 1_1 1_1 - 1_1 1_1 1_1 1_1 - 1_1$
	$1_1 1_1 1_1 - - - 1_1 1_1 - 1_1 - 1_1 1_1 1_1$				N=55 (2 sol.) F=8.85 $F_{sth}=9.47$ $F_{sth}^*=7.70$
	N=27	F=9.85	$F_{sth}=7.03$	$F_{sth}^*=5.81$	$1_1 1_1 1_1 1_1 1_1 - 1_1 - 1_1 1_1 - 1_1 - 1_1 1_1 - - - 1_1 1_1 1_1 1_1$
(2)	$*1_1 1_1^* - - - 1_1^* 1_1^* 1_1 1_1^* 1_1 - 1_1 -$				$1_1 1_1 - 1_1 1_1 - 1_1 1_1 1_1 1_1 - - - 1_1 1_1 - - - 1_1 1_1 1_1 1_1 - 1_1$
	N=29 (2 sol.)	F=6.78	$F_{sth}=7.28$	$F_{sth}^*=6.01$	$1_1 1_1 - 1_1 1_1 - 1_1 1_1 1_1 1_1 - - - 1_1 1_1 - - - 1_1 1_1 1_1 1_1 - 1_1$
	$1_1 1_1 - 1_1 1_1 - - - 1_1 1_1 - 1_1 - 1_1 1_1$				N=57 F=8.64 $F_{sth}=9.59$ $F_{sth}^*=7.78$
(2)	$*1_1 1_1^* 1_1^* - 1_1^* - - - 1_1^* 1_1^* 1_1 - 1_1$				$1_1 1_1 - 1_1 - 1_1 - - - 1_1 1_1 - 1_1 1_1 - - - 1_1 - 1_1 - 1_1 1_1 - 1_1 1_1$
	N=31	F=6.08	$F_{sth}=7.52$	$F_{sth}^*=6.20$	N=59 F=8.49 $F_{sth}=9.69$ $F_{sth}^*=7.86$
(3)	$*1_1 1_1 1_1 - 1_1 - 1_1 1_1^* - - - 1_1^* 1_1^* 1_1 1_1 -$				$1_1 1_1 1_1 1_1 - - - 1_1 - - - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 - 1_1 1_1 - 1_1 1_1$
	N=33 (2 sol.)	F=6.19	$F_{sth}=7.74$	$F_{sth}^*=6.37$	

definite recommendations for extensions of this work appear difficult. A group-theoretical approach to the problem of minimal E appears intriguing, and perhaps the use of the sieves discussed here will lead to higher F values even if not to optimal ones. But one point appears probable: if we insert $N = 750$ in (9) we obtain $F_{th} = 14.083$, which is the F -value for the Barker sequence of length 13. Since it appears likely that the use of the sieves suggested here will lead to fairly good F -values but not to optimal ones, it also appears likely that although the existence of sequences with $F > 14.083$ is conceivable, they might be forever shrouded.

A last suggestion would be the computation of the F values for the excellent Boehmer [1] sequences to determine whether Boehmer's approach could also lead to near-optimal sequences.

A concluding personal remark: Several years ago the author suggested in [4] a computer search for CS's of length 26, and this suggestion led eventually in a curious roundabout way to the discovery of a solution. Here again the hope is expressed that group-theoretical or computer-aided researchers with an interest in the intriguing

problem discussed above will have the curiosity to tackle some of the tasks just suggested; the author will be happy to act as clearing house and give information as to who is trying what. As to the computer program used to determine the sequences of Table III, the collaboration of Duncan Harris is gratefully acknowledged, and reference is made to him for any information on his program.

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